Escape rates for noisy maps

Peter Reimann* and Peter Talkner
Paul Scherrer Institut, 5232 Villigen, Switzerland
(Received 21 October 1994)

A one-dimensional bistable map in the presence of multiplicative Gaussian white noise is considered. An exact expression for the escape rate for asymptotically small noise strengths is derived consisting of an exponentially leading Arrhenius factor and a pre-exponential factor that shows a nontrivial dependence on the noise strength. The basic ingredients are a WKB ansatz for the invariant density and a discrete-time version of Kramers's flux-over-population method for the determination of the rate. In the particular case of a piecewise linear map with additive noise, the general rate formula takes a simple closed form and compares very well with numerical results.

PACS number(s): 05.40.+j, 02.50.−r

I. INTRODUCTION

The noise induced escape of a system from a locally stable state is of great importance in many areas of physics, chemistry, and biology [1–3]. For thermally activated processes, both analytical and numerical methods exist, allowing the determination of rates [1,4,5]. It is the property of detailed balance that then simplifies rate calculations considerably. In nonequilibrium systems, detailed balance is, however, in general violated with the consequence that the determination of the invariant density already becomes a nontrivial problem. For example, in the asymptotic limit of weak noise, the invariant density, which is of particular importance for the rates, may show various nonanalytic features that resemble, e.g., the behavior of the free energy of a system with a first-order phase transition or caustics in wave propagation [6,7]. Recent studies of two-dimensional Fokker-Planck systems with caustics have shown that such singularities may drastically influence the rates and lead to different results than one would expect for systems with detailed balance [8,9]. In contrast to one-dimensional Fokker-Planck processes, one-dimensional noisy systems in discrete time typically lack the property of detailed balance [10]. As a consequence, no closed analytical expressions are known in general for the invariant densities of such processes. However, in the vicinity of stable and unstable fixed and periodic points of the noiseless dynamics, the asymptotic behavior of the invariant density can be investigated in the limit of weak noise by analytical means, while for the global analysis effective numerical methods exist [11,12].

Rates of escape from locally stable states have been determined by the same methods for processes in discrete time as for continuous time processes. For example, the flux-over-population method [13,14] and the method of mean first passage time [15] have been applied to noisy maps that only weakly deviate from a continuous time system [16,17]. The Arrhenius factor, which represents the exponentially leading weak noise contribution to the rate, has been obtained by means of the most probable path that connects the relevant locally stable state with its basin boundary [18,19]. The basic idea of the reactive flux method [20] was used for piecewise linear maps for which the rate including all algebraic corrections in the noise strength was found analytically from the decay of a convenient initial state [21]. By means of this method, decay rates of point attractors, strange attractors, and strange repellers have been determined.

The present paper continues the study [10] of simple one-dimensional maps that are disturbed by weak Gaussian white noise. We utilize the flux-over-population method and apply it to the escape from a locally stable fixed point across an unstable fixed point. After a short description of the used model in Sec. II, this method is described in Sec. III. To apply it one must find a flux-carrying invariant density that matches the fluxless invariant density on the side of the initial stable state and vanishes on the opposite side. A WKB approximation of the fluxless invariant density is reviewed in Sec. IV. In Sec. V the flux-carrying density is constructed and in Sec. VI the central result for the rate is given in terms of the noise strength and a few other quantities, one part of which depends only on local properties and the other one on global properties of the noisy map. For a piecewise linear map with additive noise, these quantities are evaluated analytically and the resulting rate is compared to the findings of Ref. [21] and numerical results. The paper closes with conclusions.

II. MODEL

We study the one-dimensional dynamics of a particle with coordinate x in discrete time n, which is given by the combined action of a deterministic map $f(x)$ and a multiplicative random perturbation

$$x_{n+1} = f(x_n) + [D(x_n)]^{1/2}\,\xi_n \;, \tag{2.1}$$

*Present address: Limburgs Universitair Centrum, 3590 Diepenbeek, Belgium.

where ξ_n denotes identically distributed Gaussian white noise with density

$$P(\xi_n) = (\pi\epsilon)^{-1/2} \exp\{-\xi_n^2/\epsilon\} \ . \tag{2.2}$$

Since the random part of the dynamics (2.1) is supposed to be weak, the strength of the noise ϵ is required to be small and the noise coupling function $D(x)$ to be bounded. For the sake of convenience we assume symmetry of the noisy dynamics (2.1) under $x \mapsto -x$, implying $f(-x) = -f(x)$ and $D(-x) = D(x)$. Further, the noise coupling function $D(x)$ is assumed to be smooth and everywhere positive. Finally, the map $f(x)$ is required to be smooth and to have three fixed points, an unstable one at $x_u = 0$ and two stable ones at $\pm x_s$ with $\mathbb{R}_\pm$ as respective basins of attraction. Accordingly, the slopes at the fixed points are restricted by $f'(0) > 1$ and $|f'(\pm x_s)| < 1$.

In the stationary state, the dynamics (2.1) is governed by a unique invariant density $W(x)$ obeying the master equation

$$W(x) = \int_{-\infty}^{\infty} P(x|y)\, W(y)\, dy \ , \tag{2.3}$$

where $P(x|y)$ denotes the transition probability for a particle to move from y to x in one time step. From the noisy dynamics (2.1) and the probability density of the noise (2.2) one finds that

$$P(x|y) = [\pi\epsilon D(y)]^{-1/2} \ \exp\left\{-\frac{[x-f(y)]^2}{\epsilon\, D(y)}\right\} \ . \tag{2.4}$$

From the symmetry of $f(x)$ and $D(x)$ it follows that the density $W(x)$ is an even function of x.

III. FLUX-OVER-POPULATION METHOD

In this section we briefly describe the flux-over-population method, which has been introduced for continuous-time systems by Farkas [13] and Kramers [14] and has been generalized to the case of discrete time in Ref. [16]. In order to determine the rate at which particles go from one stable fixed point to the other, say, from x_s to $-x_s$, one constructs a flux-carrying density $\rho(x)$. To compensate for particles that escape from $\mathbb{R}_+$ to $\mathbb{R}_-$, new particles must be injected in $\mathbb{R}_+$ and, at the same time, arriving particles must be removed from $\mathbb{R}_-$. Therefore, $\rho(x)$ is the solution of the following modified master equation:

$$\rho(x) = \int_{-\infty}^{\infty} P(x|y) a(y) \rho(y)\, dy + S(x), \tag{3.1}$$

where $S(x)$ denotes the density of sources, which are located at positive values of x, and $1-a(y)$ is the absorption probability at the point y, given y is visited; for positive values of y there is no absorption, i.e., $a(y) = 1$, while for negative ones particles survive only with probability $a(y) \leq 1$. The net flux J from $\mathbb{R}_+$ to $\mathbb{R}_-$ is given by the number of particles $\int_0^\infty S(x)\, dx$ that are injected into $\mathbb{R}_+$ at each time step. A rate k can then be defined as the flux J divided by the population N of $\mathbb{R}_+$:

$$k := \frac{J}{N} = \frac{\int_0^\infty S(x)\, dx}{\int_0^\infty \rho(x)\, dx}. \tag{3.2}$$

This ratio coincides with the rate constant at which particles jump in the fluxless stationary state across the unstable fixed point from left to right and vice versa, if all those particles that are injected into $\mathbb{R}_+$ equilibrate first within the vicinity of x_s before they escape. Furthermore, particles must be removed from $\mathbb{R}_-$ only if they are outside of the so-called barrier region, which is situated around the unstable fixed point $x_u = 0$ and extends a few $l_\epsilon = (\epsilon \,/\{[f'(0)]^2 - 1\})^{1/2}$ in both positive and negative directions. For points outside this region, the probability of reaching the unstable fixed point x_u is exponentially small. According to the first condition, which requires that only equilibrated particles escape, the source term $S(x)$ must vanish within the barrier region and according to the second condition the survival probability $a(y)$ must be unity within the barrier region. In other words, the number of particles is conserved in the barrier region and, consequently, there the flux-carrying density $\rho(x)$ obeys the same equation as does the fluxless invariant density $W(x)$:

$$\rho(x) = \int_{-\infty}^{\infty} P(x|y)\, \rho(y)\, dy \ \text{ for } x \text{ in the barrier region.} \tag{3.3}$$

Following the flux-over-population method, we do not directly solve the master equation (3.1) with prescribed sources and sinks. Instead we construct a flux-carrying density that solves Eq. (3.3) in the barrier region and matches the invariant density $W(x)$ at positive values of x:

$$\rho(x) = W(x) \text{ for sufficiently large } x. \tag{3.4}$$

For negative values of x the flux-carrying density is required to be negligibly small compared to the fluxless invariant density $W(x)$:

$$\rho(x) = 0 \text{ for sufficiently large negative } x. \tag{3.5}$$

Once $\rho(x)$ is known, the density of sources $S(x)$ readily follows from the master equation (3.1). It yields, with (3.2) for the rate [16,12],

$$k = \frac{\left[\int_{-\infty}^0 dx \int_0^\infty dy - \int_0^\infty dx \int_{-\infty}^0 dy\right] \ P(x|y)\, \rho(y)}{\int_0^\infty \rho(x)\, dx} \ . \tag{3.6}$$

The first double integral in the numerator represents the flux of particles crossing the unstable fixed point from $\mathbb{R}_+$ into $\mathbb{R}_-$, while the second one accounts for the backflow from $\mathbb{R}_-$ into $\mathbb{R}_+$.

To summarize, we have to determine in a first step the invariant density $W(x)$ in $\mathbb{R}_+$ from the master equation (2.3), then the flux-carrying density $\rho(x)$ as a solution of Eq. (3.3), which approaches the required behavior (3.4) and (3.5) outside the barrier region, and finally the escape rate k according to (3.6). This will be carried out in Secs. IV, V, and VI, respectively. Finally, we note that also the survival probability $a(y)$ can be determined from the master equation (3.1) once $\rho(x)$ is known, though the explicit forms of both $S(x)$ and $a(y)$ are not needed.

IV. INVARIANT DENSITY

In this section we briefly summarize a WKB approach for the determination of the invariant density $W(x)$ from Eq. (2.3); see Refs. [10,12] for details. Since $W(x)$ is needed for positive values of x only [see Eq. (3.4)] we restrict ourselves to $x \geq 0$ in this section. We start with a WKB ansatz for the invariant density

$$W(x) = \epsilon^{-1/2}\, Z_\epsilon(x)\, e^{-\phi(x)/\epsilon} , \tag{4.1}$$

where $\phi(x)$ is the generalized potential, $Z_\epsilon(x)$ is the prefactor, and the factor $\epsilon^{-1/2}$ has been extracted for the sake of convenience. The generalized potential is required to obey the functional equation

$$\phi(x) = \min_y \{\phi(y) + [x - f(y)]^2/D(y)\} . \tag{4.2}$$

Inserting the ansatz (4.1) and the transition probability (2.4) into the master equation (2.3), one finds the following integral equation for the prefactor:

$$Z_\epsilon(x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{\pi\epsilon D(y)}}\, Z_\epsilon(y)\, e^{-V(x,y)/\epsilon} , \tag{4.3}$$

where

$$V(x,y) := \phi(y) - \phi(x) + [x - f(y)]^2/D(y) . \tag{4.4}$$

Note that in contrast to the generalized potential following from Eq. (4.2), the prefactor $Z_\epsilon(x)$ may still depend on the noise strength ϵ. In the following two subsections we discuss how the generalized potential and the prefactor are obtained from the functional equation (4.2) and the integral equation (4.3), respectively.

A. Generalized potential

The value of y that minimizes the right-hand side of the functional equation (4.2) for a given value of x is denoted by $g(x)$. It can be shown that Eq. (4.2) has a uniquely defined solution $\phi(x)$ under the following conditions: (i) $\phi(x)$ vanishes at the stable fixed point x_s; (ii) for each $x > 0$ the series of iterates of $g(x)$ converges to the stable fixed point x_s, i.e., $g^i(x) \to x_s$ for $i \to \infty$ and $x > 0$. The explicit construction of this solution is equivalent to the determination of the separatrix in a nonintegrable Hamiltonian system. Closed analytical expressions for $\phi(x)$ are thus available only for special maps $f(x)$ and noise coupling functions $D(x)$. For an example see Sec. VI. For an efficient numerical method we refer to [10,11], while an analytical approximation scheme is given in [12]. The qualitative features of the generalized potential $\phi(x)$ and the function $g(x)$ are nevertheless well understood. For example, at the stable fixed point $x = x_s$ the generalized potential has a global minimum and $g(x)$ considered as a map has a stable fixed point. The functions $\phi(x)$ and $g(x)$ possess series expansions about the fixed point x_s, which read, in leading order in $(x - x_s)$,

$$\phi(x) = (x - x_s)^2 \{1 - [f'(x_s)]^2\}/D(x_s), \tag{4.5}$$

$$g(x) = x_s + (x - x_s)\, f'(x_s) . \tag{4.6}$$

These series expansions represent the functions $\phi(x)$ and $g(x)$ up to a point b_0 with $0 < b_0 < x_s$. In the interval $[0, b_0]$ the generalized potential $\phi(x)$ consists of the minimal branches of a family of smooth functions $\phi_i(x)$, $i = 1, 2, \ldots$,

$$\phi(x) = \min_i \phi_i(x) ; \tag{4.7}$$

see also Fig. 1. Each function contributes only on a single interval I_i to the generalized potential:

$$\phi(x) = \phi_i(x) \text{ for } x \in I_i . \tag{4.8}$$

The label i can consequently be chosen such that with increasing values of i the values of $x \in I_i$ decrease. The boundary points b_i of the intervals $I_i = [b_i, b_{i-1})$ then coincide with the intersection points of $\phi_{i+1}(x)$ and $\phi_i(x)$, which are unique in $[0, b_0]$. Beyond the point b_0 the generalized potential continues smoothly and approaches the form (4.5) near the stable fixed point x_s. For large i, corresponding to small positive values $x \in I_i$, the functions $\phi_i(x)$ approach straight lines,

$$\phi_i(x) = \bar{\phi}(x) - \frac{1}{2}\{x - a^*[f'(0)]^{-i}\}^2 \bar{\phi}''(0) \quad \text{for } x \in I_i, \tag{4.9}$$

which touch the parabola

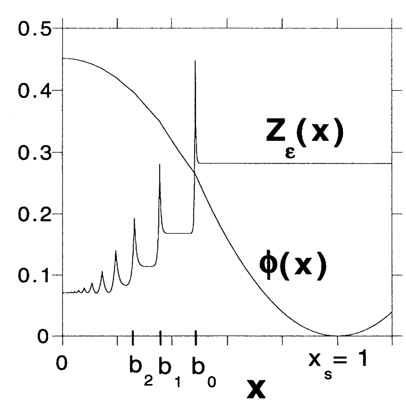

FIG. 1. The generalized potential $\phi(x)$ and the prefactor $Z_\epsilon(x)$ for the piecewise linear map (7.1) with parameter values $s = 0$, $u = 1.35$, $x_s = 1$, and $\epsilon = 0.001$. For larger x values the generalized potential continues to increase quadratically and $Z_\epsilon(x)$ stays constant. Both functions are symmetric about $x = 0$.

$$\bar{\phi}(x) = \phi(0) - \frac{[f'(0)]^2 - 1}{D(0)} x^2 \tag{4.10}$$

at the points $a^*[f'(0)]^{-i}$ accumulating at $x_u = 0$. The quantities a^* and $\phi(0)$ are both positive and depend on $f(x)$ and $D(x)$ in a global way. In general, their values cannot be determined analytically. For large i the approximation (4.9) yields, for the intersection points,

$$b_i = \frac{f'(0) + 1}{2[f'(0)]^{i+1}} a^* \ . \tag{4.11}$$

The function $g(x)$ is also smooth on the intervals I_i, but in contrast to $\phi(x)$ has discontinuities at the boundary points b_i. Within the same approximation that yields Eq. (4.9) $g(x)$ becomes piecewise linear

$$g(x) = \frac{x}{f'(0)} + \frac{[f'(0)]^2 - 1}{[f'(0)]^{i+1}} a^* \text{ for } x \in I_i \tag{4.12}$$

for small positive x or, equivalently, for large i.

One can easily verify that Eqs. (4.5) and (4.6) are solutions of the functional equation (4.2) for the linear map

$$f(x) = x_s + f'(x_s)(x - x_s) \tag{4.13}$$

and constant noise coupling function

$$D(x) = D(x_s). \tag{4.14}$$

Similarly, Eqs. (4.8), (4.9), and (4.12) solve the functional equation (4.2) for

$$f(x) = f'(0)x \tag{4.15}$$

and

$$D(x) = D(0). \tag{4.16}$$

We want to stress that though the global solution of the functional equation (4.2) supplemented by the above stated conditions (i) and (ii) as well as the local solution near the stable fixed point are uniquely defined, the precise form of the generalized potential near the unstable fixed point cannot be found alone from the local properties of the noisy map at the unstable fixed point. For example, the sequence of boundary points b_i is determined by the parameter a^*, which depends in a global way on the noisy map restricted to the domain of attraction of the fixed point x_s [10,12]. In general, no analytical means are available to single out the particular local solution of the approximating unstable linear map with additive noise that matches the generalized potential of the full nonlinear noisy map. The generalized potential $\phi(x)$ is shown in Fig. 1 for a piecewise linear map with additive noise as defined in Eq. (7.1) below. The discontinuities of the slope of $\phi(x)$ at the points b_i are not very pronounced, but still visible in this plot.

B. Prefactor

Next we discuss the prefactor $Z_\epsilon(x)$ following from the integral equation (4.3). The functional equation (4.2) implies that $V(x,y)$ in (4.4), considered as a function of y, takes its minimal value zero at $y = g(x)$. A saddle point approximation of the integral equation (4.3) then yields

$$Z_\epsilon(x) = \left(\frac{1}{2} D(g(x)) \frac{\partial^2}{\partial y^2} V(x, y = g(x))\right)^{-1/2} Z_\epsilon(g(x)) \ . \tag{4.17}$$

This approximation is justified for small noise strengths ϵ such that the integral in (4.3) is dominated by a small neighborhood of the minimum of $V(x,y)$ at $y = g(x)$ in which $V(x,y)$ is smooth and variations of $Z_\epsilon(x)$ can be neglected. The latter condition turns out to be always satisfied, while cases in which the former ones are violated are discussed below.

We first consider a vicinity of the stable fixed point x_s. Using in Eqs. (4.4) and (4.17) the approximate expressions (4.5), (4.6), (4.13), and (4.14) one obtains a smooth prefactor $Z_\epsilon(x)$, which factorizes into an ϵ- and an x-independent part. The ϵ-dependent factor is fixed by the total normalization of the invariant density $\int_{-\infty}^{\infty} W(x)dx = 1$. Using the WKB ansatz (4.1) for small noise strengths ϵ, the normalization integral can be evaluated in saddle point approximation about the stable fixed points $\pm x_s$. This yields, for the prefactor at x_s,

$$Z_\epsilon(x_s) = \sqrt{\frac{1 - [f'(x_s)]^2}{4\pi D(x_s)}} \ . \tag{4.18}$$

Consequently, $Z_\epsilon(x)$ is ϵ independent in a whole neighborhood of x_s. Since the iterates of $g(x)$ converge towards x_s, the solution for the prefactor near x_s may successively be continued to larger values of $|x - x_s|$ by means of (4.17). Within the validity of the saddle point approximation on which Eq. (4.17) is based the prefactor is thus completely independent of the noise strength ϵ.

Close to the unstable fixed point $x_u = 0$, Eq. (4.17) simplifies with Eqs. (4.8), (4.12), (4.15), and (4.16) to read $Z_\epsilon(x) = Z_\epsilon(g(x))/f'(0)$. For the derivative one finds that $Z'_\epsilon(x) = Z'_\epsilon(g(x))/[f'(0)]^2$. Hence, in the limit $x \to 0$ the derivative $Z'_\epsilon(x)$ approaches zero much faster than $Z_\epsilon(x)$. Therefore, the prefactor asymptotically assumes constant values on the intervals I_i:

$$Z_\epsilon(x) = Z^* \, [f'(0)]^{-i} \quad \text{for } x \in I_i, \tag{4.19}$$

where Z^* is an ϵ-independent positive quantity, which, like a^*, depends globally on $f(x)$ and $D(x)$ and is in general not known in closed analytical form. Note that the example shown in Fig. 1 is somewhat atypical because the steps of the prefactor are of constant height also for large values of x. We finally mention that for x values within a neighborhood of a point b_i with an extension of a few ϵ/b_i, a second minimum of $V(x,y)$ exists besides the one at $y = g(x)$ that contributes also notably to the integral in Eq. (4.3). As a consequence, on top of the steps of $Z_\epsilon(x)$ at the points b_i, cusps are superimposed that are of the height $Z^*([f'(0)]^{-i} + [f'(0)]^{-i-1})$ and width of order ϵ/b_i; see Fig. 1. For $b_i = O(\sqrt{\epsilon})$ these cusps merge. For values of x of the same order of magnitude, also $V(x,y)$ is no longer a smooth function of y within a sufficiently large neighborhood of $g(x)$. Therefore, the saddle point approximation leading to Eq. (4.17) and its solution (4.19) are no longer valid for $x \leq O(\sqrt{\epsilon})$. For small but finite noise strengths ϵ the proper behavior of $Z_\epsilon(x)$ in this region will be found as a by-product in the next section. On the other hand, in the weak noise limit $\epsilon \to 0$, Eqs. (4.17) and (4.19) are valid for all positive values of x with the exception of the interval boundaries b_i. In particular, the constant Z^* may be obtained in this limit as

$$Z^* = \lim_{i\to\infty} \lim_{\epsilon\to 0} [f'(0)]^i Z_\epsilon(a^*[f'(0)]^{-i}). \tag{4.20}$$

We finally mention that the singularities of $\phi(x)$ and $Z_\epsilon(x)$ at the points b_i compensate each other in the WKB ansatz (4.1) and result for finite noise strengths ϵ in a smooth invariant density $W(x)$.

V. FLUX-CARRYING DENSITY

In order to find a flux-carrying density $\rho(x)$ that solves Eq. (3.3) and satisfies the matching conditions (3.4) and (3.5) we approximate the transition probability (2.4) for small values of y by means of the Eqs. (4.15) and (4.16), yielding

$$P(x|y) = [\pi\epsilon D(0)]^{-1/2} \exp\left\{-\frac{[x - f'(0)y]^2}{\epsilon\, D(0)}\right\} . \tag{5.1}$$

Using this approximation for $P(x|y)$ and the linear approximation (4.9) for $\phi_i(x)$, a simple calculation shows that

$$\rho(x) = \epsilon^{-1/2} Z^* \sum_{i=-\infty}^{\infty} [f'(0)]^{-i}\, e^{-\phi_i(x)/\epsilon} \tag{5.2}$$

solves Eq. (3.3). Note that the sum on the right-hand side converges for all $x \in \mathbf{R}$. Putting Eq. (5.2) back into the full master equation (3.3) containing the exact transition probability (2.4), one finds that for small values of x only small values of y contribute notably to the integral in (3.3). This demonstrates the consistency of the approximation (5.1) and the solution (5.2).

Next we consider small positive x values in the interior of an interval I_{i_0}, $i_0 \gg 1$, which keep a minimal distance of a few ϵ/b_{i_0} from the boundaries b_{i_0} and b_{i_0-1}. According to (4.11) this is possible for values of x that are larger than a few $l_\epsilon = (\epsilon/\{[f'(0)]^2 - 1\})^{1/2}$. For such values of x, only the term with $i = i_0$ contributes notably to the sum in (5.2). Under the additional condition that x is contained in a small but ϵ-independent neighborhood of $x_u = 0$ where the approximations (4.9) and (4.19) are valid, we find that $\rho(x)$ in (5.2) coincides with $W(x)$ in (4.1). For values of x that are in the same range but closer to a boundary point of I_{i_0}, say to b_{i_0}, than a few ϵ/b_{i_0}, the two terms with $i = i_0$ and $i = i_0+1$ contribute notably to the sum in (5.2). Again, this reproduces $W(x)$ as given by Eq. (4.1) including the correct cusp of $Z_\epsilon(x)$ at b_{i_0} as mentioned below Eq. (4.19). Hence the expression (5.2) solves the master equation (3.3) in the barrier region and approaches the fluxless invariant density $W(x)$ according to Eq. (3.4).

For negative values of x as well as for positive ones that are smaller than a few l_ϵ, many terms contribute notably to the sum (5.2), which can then be approximated reasonably well by an integral over i extending over the real axis. With Eq. (4.9) this leads to

$$\rho(x) \simeq \frac{Z^* e^{-\bar\phi(x)/\epsilon}}{2a^* \ln f'(0)} \sqrt{\frac{\pi D(0)}{[f'(0)]^2 - 1}} \times \mathrm{erfc}\left(-\sqrt{\frac{[f'(0)]^2 - 1}{\epsilon D(0)}}\, x\right) , \tag{5.3}$$

where $\mathrm{erfc}(z) = 2\pi^{-1/2}\int_z^\infty e^{-y^2} dy$ is the complementary error function. Using the asymptotic form $\mathrm{erfc}(z) \simeq e^{-z^2}/\sqrt{\pi}z$, valid for large values of z, the asymptotic behavior of $\rho(x)$ for sufficiently large negative values of x becomes

$$\rho(x) \simeq \frac{D(0)}{f'(0)^2 - 1} \frac{Z^* e^{-\bar\phi(0)/\epsilon}}{2a^* \ln f'(0)} \frac{\epsilon}{|x|} . \tag{5.4}$$

Therefore, $\rho(x)$ approaches zero for negative x as required by Eq. (3.5). In summary, satisfying Eqs. (3.3)–(3.5), $\rho(x)$ is a proper flux-carrying invariant density.

Finally, we return to the invariant fluxless density, which is yet undetermined near the unstable fixed point $x = 0$ for finite noise. For a symmetric noisy map, i.e., one with $f(x) = -f(-x)$ and $D(x) = D(-x)$, the sum $\rho(x) + \rho(-x)$ solves the master equation (3.3) in the barrier region and according to the Eqs. (3.4) and (3.5) approaches the fluxless invariant density $W(x)$ following from the WKB approximation on both the positive and the negative side of the unstable fixed point. Hence the sum represents the invariant fluxless density for all values of x,

$$W(x) = \rho(x) + \rho(-x) . \tag{5.5}$$

Together with the WKB ansatz (4.1) and the local solution (5.2) for $\rho(x)$, this expression for $W(x)$ allows one to determine the prefactor $Z_\epsilon(x)$ for those small values of x for which the saddle point approximation of Eq. (4.3) does not hold. In particular, at the unstable fixed point $x_u = 0$ one finds that

$$Z_\epsilon(0) = \kappa\left(f'(0),\ \frac{a^{*2}}{\epsilon D(0)}\right) \sqrt{\frac{\pi\epsilon D(0)}{[f'(0)]^2 - 1}} \frac{Z^*}{a^* \ln f'(0)} , \tag{5.6}$$

where

$$\kappa(u,v) = \frac{\sum_{i=-\infty}^{\infty} u^{-i} \exp\{-[u^2-1]u^{-2i}v\}}{\int_{-\infty}^{\infty} di\, u^{-i} \exp\{-[u^2-1]u^{-2i}v\}}$$
$$= 2\sqrt{[u^2-1]v/\pi}\ln u \times \sum_{i=-\infty}^{\infty} u^{-i} \exp\{-[u^2-1]u^{-2i}v\} \ . \quad (5.7)$$

Note that $\kappa(u, u^{2i}v) = \kappa(u,v)$ for all integer i. Closer inspection shows that $\kappa(u,v) \to 1$ for $u \to 1$ and arbitrary values of v. For very small values of x the prefactor remains constant with minute maxima superimposed at the points $\pm b_i$ which accumulate at $x_u = 0$; see Fig. 1.

VI. ESCAPE RATE

The population $N = \int_0^\infty \rho(x)dx$ in the denominator of the rate formula (3.6) is readily determined. The flux-carrying density $\rho(x)$ is concentrated about the stable fixed point x_s, where it coincides with the fluxless invariant density $W(x)$. Since this is symmetric about $x = 0$ and normalized on the real axis, one finds that $N = 1/2$. The flux J in the numerator of the rate formula (3.6) is given in terms of integrals that get their main contribution from the barrier region. It is therefore sufficient to replace the transition probability entering these integrals by the approximate expression (5.1). Details of the calculation are given in the Appendix. For the rate one obtains

$$k = \sqrt{\epsilon}\, \frac{Z^*}{a^*}\, \frac{D(0)}{[f'(0)]^2 - 1}\, e^{-\phi(0)/\epsilon}, \quad (6.1)$$

which is the central result of this paper. It is asymptotically exact for small noise strengths ϵ. For a comparison with numerical data see Fig. 2.

There are two globally defined, ϵ-independent quantities $\phi(0)$ and Z^*/a^* that enter the rate (6.1). The first one represents the difference of the generalized potential at the unstable and the stable fixed points of the deterministic map $f(x)$ and together with the noise strength ϵ determines the exponentially leading part of the rate, which corresponds to the Arrhenius factor in a thermally activated process. The second quantity Z^*/a^* characterizes the prefactor near the unstable fixed point $x_u = 0$ in the limit of weak noise according to (4.20). In contrast to rates that follow from a one-dimensional Smoluchowski equation in continuous time [1], the pre-exponential factor of the rate (6.1) depends on the noise strength.

Using the expressions (4.5) and (4.10) for the generalized potential and Eqs. (4.18) and (5.6) for the prefactor, the rate (6.1) may be rewritten as

$$k = \kappa\left(f'(0),\ \frac{a^{*2}}{\epsilon D(0)}\right)^{-1} \frac{\lambda}{2\pi} \sqrt{\frac{\phi''(x_s)}{|\bar{\phi}''(0)|}}\, \frac{W(0)}{W(x_s)}\ , \quad (6.2)$$

where $\lambda = \ln f'(0)$ represents the local Lyapunov exponent of the map $f(x)$ at $x = 0$. In this form, the rate

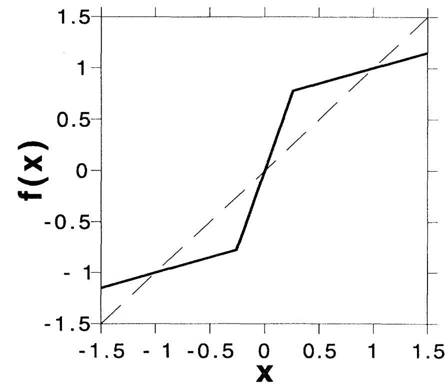

FIG. 2. The piecewise linear map from Eq. (7.1) with $s = 0.3$, $u = 3$, and $x_s = 1$. The dashed line represents the identical map.

resembles the Langer formula [23] except that it contains the factor κ and that the curvature of the potential at the unstable fixed point, which does not exist in the present case, is replaced by $\bar{\phi}''(0)$.

VII. PIECEWISE LINEAR MAP

We consider a piecewise linear map with additive noise:

$$\begin{aligned} D(x) &= 1 && \text{for all } x, \\ f(x) &= x_s + s(x - x_s) && \text{for } x \geq x_m, \\ f(x) &= u\,x && \text{for } 0 \leq x \leq x_m\ , \end{aligned} \quad (7.1)$$

where x_m is the matching point of the two linear pieces $x_m = x_s(1-s)/(u-s)$. For an example see Fig. 2. The behavior of the map for negative x follows from the symmetry $f(-x) = -f(x)$. The slopes s and u at the fixed points obey $u > 1$ and $0 \leq s < 1$, where the restriction $s \geq 0$ guarantees that $\mathbb{R}_+$ and $\mathbb{R}_-$ are the basins of attractions of x_s and $-x_s$, respectively.

By means of a straightforward but somewhat tedious calculation one can verify that the functional equation (4.2) is solved by a generalized potential of the form (4.8) with

$$\phi_i(x) = \frac{(u^2-1)(1-s^2)}{u^{2i}(u^2-s^2) - 1 + s^2}\,(x - u^i x_s)^2 \quad (7.2)$$

for $x \geq 0$ and $\phi(x) = \phi(-x)$ for $x < 0$; see Fig. 1. In particular, $\phi_0(x)$ represents the series expansion of $\phi(x)$ about the unstable fixed point x_s, which is valid for all $x \in [b_0, \infty)$. We recall that b_i is the intersection point of $\phi_i(x)$ and $\phi_{i+1}(x)$ in $[0, b_0]$. For large i b_i reads

$$b_i = \frac{(1-s^2)(u+1)}{2(u^2-s^2)u^{i+1}}\, x_s. \quad (7.3)$$

Hence, for large i and $x \in I_i$ the terms in Eq. (7.2) that are quadratic in x can be neglected. With the resulting expression one recovers Eqs. (4.9) and (4.10) with

$$a^* = x_s\,(1-s^2)/(u^2-s^2), \tag{7.4}$$
$$\phi(0) = x_s^2\,(u^2-1)(1-s^2)/(u^2-s^2)\ . \tag{7.5}$$

For the function $g(x)$ one exactly retrieves Eqs. (4.6) and (4.12) on the intervals $[b_0,\infty)$ and I_i, $i \geq 1$, respectively. In particular, $g^i(x)$ tends to x_s for large i and any $x > 0$ and hence together with $\phi(x_s) = 0$ fulfills the conditions guaranteeing a uniquely defined solution of the functional equation (4.2), which are oulined in the first paragraph of Sec. IV A. A more constructive approach to obtaining the generalized potential is described in [12].

Inserting the above results for $\phi(x)$ and $g(x)$ into Eq. (4.17), one obtains the following equation for the prefactor valid in saddle point approximation:

$$Z_\epsilon(x) = [\phi''(x)/\phi''(g(x))]^{1/2}\, Z_\epsilon(g(x))\ . \tag{7.6}$$

Note that $\phi''(x)$ is constant on each interval I_i. Closer inspection shows that $\phi(x)$ and $g(x)$ are smooth at the matching point x_m and that a complete neighborhood of x_m is not contained in the range Im$[g]$ of $g(x)$. Therefore, there are no further restrictions of the validity of the saddle point approximation (7.6) due to the nondifferentiability of $f(x)$ at $x = x_m$ in addition to those mentioned in Sec. IV B. Since $g(x)$ maps I_0 into itself and $g^i(x) \to x_s$ the saddle point approximation (7.6) yields a constant prefactor $Z_\epsilon(x)$ on I_0. Taking into account Eq. (4.18) and $g^i(x) \to x_s$ the iteration of Eq. (7.6) yields $Z_\epsilon(x) = \sqrt{\phi''(x)/2\pi}$. With (4.8) and (7.2) it follows that

$$Z_\epsilon(x) = u^{-i}\sqrt{\frac{1}{4\pi}\frac{(u^2-1)(1-s^2)}{u^2-s^2-(1-s^2)u^{-2i}}}\quad \text{for } x \in I_i\ . \tag{7.7}$$

Within the validity of the saddle point approximation (4.17), i.e., except for neighborhoods of the interval boundaries b_i with the extension of a few b_i/ϵ, we thus find a piecewise constant prefactor; see Fig. 1. A comparison with (4.19) implies that

$$Z^* = \sqrt{\phi(0)/(4\pi x_s^2)}\ , \tag{7.8}$$

where $\phi(0)$ is given in (7.5).

To summarize, in spite of the simplicity and the peculiarities of the piecewise linear map (7.1), both the generalized potential $\phi(x)$ and the prefactor $Z_\epsilon(x)$ exhibit just the typical properties predicted in Sec. IV; see Fig. 1. The only difference is that for more general noisy maps the generalized potential is not strictly parabolic on all intervals I_i and the prefactor is not constant in the interior parts of the intervals where the saddle point approximation holds.

Making use of (7.4), (7.5), and (7.8) the escape rate (6.1) takes the form

$$k = \sqrt{\epsilon/[4\pi\phi(0)]}\, e^{-\phi(0)/\epsilon}\ . \tag{7.9}$$

For sufficiently small noise strength ϵ this rate formula is

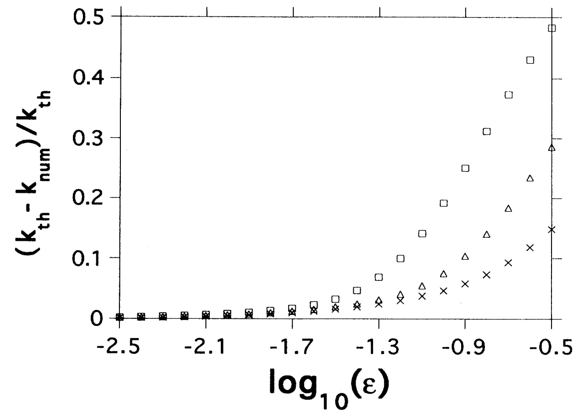

FIG. 3. The relative difference $[k_{\rm th} - k_{\rm num}]/k_{\rm th}$ of the theoretical and numerical escape rates versus $\log_{10}(\epsilon)$ for the piecewise linear map with additive noise (7.1) at the parameter values $x_s = 1$, $s = 0$, $u = 1.5$ (□), $u = 2$ (△), and $u = 4$ (×). The theoretical rate is given by (7.9). $k_{\rm num}$ is calculated from (3.6) with a numerical solution $\rho(x)$ of the master equation (3.3) with boundary conditions (3.4) and (3.5). For sufficiently small noise strengths ϵ the agreement between theory and numerics is excellent.

in excellent agreement with numerical results as shown in Fig. 3. The identical expression has been obtained by means of a reactive flux method in the limit of weak noise in [21].

Closed analytical results as in this section can be derived also for negative values of s in Eq. (7.1) or for bistable maps $f(x)$ with more than three linear pieces and piecewise constant noise-coupling functions $D(x)$.

VIII. CONCLUSIONS

In this paper we adopted the flux-over-population method of constructing rates of escape from a locally stable state across an unstable fixed point for one-dimensional systems in discrete time. The decisive quantity of this method is a flux-carrying invariant density that matches the fluxless density near the initial state and vanishes beyond the unstable fixed point. We constructed the flux-carrying invariant density explicitly and determined the escape rate from it.

In the weak noise limit this rate has an exponentially leading contribution that resembles the Arrhenius factor of a rate of a thermally activated process. Additionally there is a prefactor that decreases with the square root of the noise strength. This is unfamiliar from the point of view of one-dimensional processes in continuous time, although there are indications that such behavior may also occur for two-dimensional Fokker-Planck processes without detailed balance.

From the flux-carrying invariant density the fluxless invariant density may easily be constructed. If the latter is represented in a WKB form it reproduces the known singularities of the generalized potential governing the exponentially leading part and of the prefactor [10,11,22].

Moreover, we found analytic expressions of the prefactor in the close vicinity of the unstable fixed point that are not yet available from a WKB analysis. The value of this prefactor at the unstable fixed point decreases also with the square root of the noise strength being the reason for the same effect for the prefactor of the rate.

For the sake of simplicity, in this paper we restricted ourselves to the discussion of noisy maps that are symmetric about the unstable fixed point. This assumption may, however, easily be dropped [12]. For sufficiently weak noise the flux-carrying density decreases fast enough beyond the unstable fixed point such that there the nonlinear contributions of the map, which may break the symmetry, are of no importance. A similar argument applies for a possible asymmetry stemming from the noise coupling function $D(x)$. Consequently, the flux-carrying density given in this paper holds also for nonsymmetric noisy maps. In this way there belongs to each locally stable state a flux-carrying density. The rates of escape from these states follow from the same rate formula (6.1), where the global constants Z^*/a^* and $\phi(0)$ are different for the different rates. The fluxless invariant density is again given by a linear combination of the flux-carrying densities belonging to the different locally stable states; cf. Eq. (5.5). The corresponding weights are uniquely determined by the total normalization and the condition that the net flux vanishes at the unstable fixed point. The generalization of this method to maps with more than two stable states is straightforward. For periodic maps a fluxless invariant density, however, need not exist.

ACKNOWLEDGMENT

P. R. gratefully acknowledges financial support by the Program on Inter-University Attraction Poles of the Belgian Government and the Swiss National Science Foundation.

APPENDIX: CALCULATION OF THE FLUX J

In order to determine the flux J in the numerator of (3.6) we make use of the properties (3.4), (3.5), and (5.2), the flux-carrying density $\rho(x)$, and the behavior of the invariant density $W(x)$ as discussed in Sec. IV. In a first step we show that in the numerator of (3.6) only small x and y values contribute notably for small noise strengths ϵ. Then, in this region the approximations for the transition probability (5.1) and for the flux-carrying density (5.2) are introduced. Finally, it can be shown that these approximations can be extended over the entire domains of integration with negligible effect on the flux J. We thus find that

$$J = \frac{Z^*\, e^{-\phi(0)/\epsilon}}{\epsilon\sqrt{\pi\, D(0)}} \sum_{i=-\infty}^{\infty} \exp\left\{ -\frac{u^2-1}{\epsilon D(0)} \frac{a^{*2}}{u^{2i}} \right\} \frac{S_i}{u^i}, \tag{A1}$$

$$S_i := \left[\int_{-\infty}^{0} dx \int_{0}^{\infty} dy - \int_{0}^{\infty} dx \int_{-\infty}^{0} dy \right] \times \exp\left\{ -\frac{2a^*[u^2-1]y}{\epsilon\, D(0)\, u^i} - \frac{[x-uy]^2}{\epsilon\, D(0)} \right\}, \tag{A2}$$

where we introduced $u = f'(0)$. Equation (A2) can be recast into

$$S_i = \left[\int_{0}^{\infty} dy \int_{-\infty}^{-uy} dx - \int_{-\infty}^{0} dy \int_{-uy}^{\infty} dx \right] \times \exp\left\{ -\frac{2a^*[u^2-1]y}{\epsilon\, D(0)\, u^i} - \frac{x^2}{\epsilon\, D(0)} \right\} \tag{A3}$$

and then, by exchanging the order of integration, into

$$S_i = \left[\int_{-\infty}^{0} dx \int_{0}^{-x/u} dy - \int_{0}^{\infty} dx \int_{-x/u}^{0} dy \right] \times \exp\left\{ -\frac{2a^*[u^2-1]y}{\epsilon\, D(0)\, u^i} - \frac{x^2}{\epsilon\, D(0)} \right\} . \tag{A4}$$

Carrying out the integrals over y, we find that

$$S_i = \frac{\epsilon\, D(0)\, u^i}{2a^*[u^2-1]} \int_{-\infty}^{\infty} dx \, \exp\left\{ -\frac{x^2}{\epsilon\, D(0)} \right\} \times \left[1 - \exp\left\{ -\frac{2a^*[u^2-1]x}{\epsilon\, D(0)\, u^{i+1}} \right\} \right] . \tag{A5}$$

Then the x integration can be performed and the flux (A1) takes the form

$$J = \frac{\sqrt{\epsilon}\, D(0)\, Z^*}{2\, a^*\, [u^2-1]}\, e^{-\phi(0)/\epsilon} \sum_{i=-\infty}^{\infty} \left[\exp\left\{ -\frac{u^2-1}{\epsilon D(0)} \frac{a^{*2}}{u^{2i}} \right\} - \exp\left\{ -\frac{u^2-1}{\epsilon D(0)} \frac{a^{*2}}{u^{2(i+1)}} \right\} \right] . \tag{A6}$$

Rewriting the infinite sum as $\lim_{l\to\infty} \sum_{i=-l}^{l}$ and taking into account that $u = f'(0) > 1$, the final result for the flux becomes

$$J = \sqrt{\epsilon}\, \frac{Z^*\, e^{-\phi(0)/\epsilon}}{2\, a^*} \frac{D(0)}{[f'(0)]^2 - 1} . \tag{A7}$$

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